Achievable Rates for Conferencing Multiway Channels

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***Abstract*—A generalization of the additive Gaussian two-way channel to M users is considered. Such channels contain implicit feedback in the sense that the channel output signals observed by the different encoders are correlated. While the benefits of feedback are shown to be negligible at high SNR, for moderate SNR, feedback can play a significant role in boosting the sum-rate performance. To highlight this potential gain, the special case of the M-user multiway channel with a common output is considered. By taking insights from Kramer's Fourier MEC, a feedback strategy is introduced and shown to strictly dominate the performance of a pre-log optimal non-feedback strategy. Furthermore, an upper bound is derived to show this feedback strategy achieves the sum-rate capacity beyond a certain SNR threshold. Under per-symbol power constraints, this upper bound can be tightened to show the feedback strategy is sum-rate optimal for all SNR values.**

I. INTRODUCTION

The advent of portable wireless devices enables a kind of conferencing reminiscent of conference calls, in which each user in a group is interested in sending and receiving data to and from all members of the group. For example, members of an emergency response team, each equipped with a wireless enabled device, may want to transmit and receive data to and from all other responders at a disaster site.

This note focuses on achievable rates for such channels, which we will refer to as conferencing multiway channels. While other notions of multiway channels exist [1], [2], for the purposes of this note, an M-user multiway channel is one in which each user has a message and wants to decode the messages of all other users. The best known of these is the two-way channel, which was introduced by Shannon [3]. Shannon provided a general capacity region along with single-letter inner and outer bounds. For the additive Gaussian two-way channel, Shannon's inner and outer bounds coincide and lead to a natural coding strategy. Each user simply subtracts off its own signal, and the result is two point-to-point channels, for which block codes without feedback can be used to attain the capacity region.

We consider the performance of achievable strategies for additive Gaussian M-user multiway channels and give partial capacity results in the high signal-to-noise ratio limit. In this limiting regime, a simple strategy that does not depend on the channel outputs is sufficient to prove our results.

However, more intricate strategies that leverage the ability of each user to use their channel outputs in their code construction show promise at improving the rate. To demonstrate this benefit, we consider a special case of our model, in which all users observe a common channel output. For this case, we first consider a strategy known as Fourier MEC [5], a generalization of the feedback schemes considered by Schalkwijk and Kailath [4] and Ozarow [9]. Furthermore, it has been shown to be optimal for the M-user Gaussian MAC under a per-symbol power constraint [6].

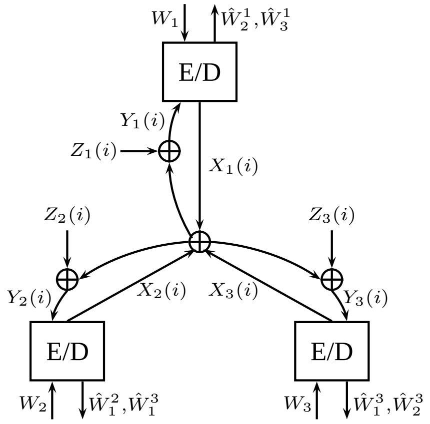

Fig. 1. A 3-user multiway channel

Unfortunately, Fourier MEC turns out to be pre-log suboptimal for the common output M-user multiway channel. We introduce a new class of strategies that combine features of Fourier MEC with those of a standard block coding strategy to construct a pre-log optimal strategy that exploits this feedback. Furthermore, we show that this strategy achieves the sum-rate capacity beyond a certain SNR threshold. Furthermore, we tighten these upper bounds under a per-symbol power constraint to show the strategy achieves the sum-rate capacity at all SNR.

II. PROBLEM SETUP

A. Notation

Assume throughout that logarithms are natural and units of information are expressed in nats per dimension. We use capital letters for X, Y, Z random variables and calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ for sets. K_X denotes the covariance matrix of

X. We further denote their determinants by

$$V_X = |K_X| \tag{1}$$

$$V_{X|Y} = \frac{|K_{X,Y}|}{|K_Y|} \ . \tag{2}$$

B. Channel Model

Consider the complex channel with complex inputs

$$Y_\ell(i) = \sum_{k=1}^{M} X_k(i) + Z_\ell(i) \ , \tag{3}$$

where the marginals $Z_\ell(i) \sim \mathcal{N}_c(0, \sigma^2)$ are complex circular Gaussian random variables independent over i, and $Z_1(i), \ldots, Z_M(i)$ have covariance matrix K_Z. Rates $(R_1, R_2, \ldots, R_M)$ are achievable if for all $\delta > 0$, there exists n_0 such that for $n \geq n_0$, complex encoding functions for $1 \leq i \leq n$,

$$X_\ell(i) = f_i^\ell(W_\ell, Y_\ell^{i-1}) \tag{4}$$

satisfying $\sum_{i=1}^n E|X_\ell(i)|^2 \leq nP$, decoding functions g_ℓ^n,

$$(\hat{W}_1^\ell, \ldots, \hat{W}_M^\ell) = g_\ell^n(W_\ell, Y_\ell^n) \ , \tag{5}$$

where messages $W_\ell \in \{1, \ldots, e^{n(R_\ell - \delta)}\}$, such that for all ℓ, ℓ'

$$P(\hat{W}_\ell^{\ell'} \neq W_\ell) \leq \delta \ . \tag{6}$$

For notational convenience, we also introduce $s = \frac{P}{\sigma^2}$, which we will refer to as the SNR.

C. Definition of pre-log capacity

We now consider a definition that is useful for understanding the performance of strategies at high SNR.

The sum-rate pre-log capacity is defined as

$$\Gamma = \lim_{P \to \infty} \frac{C_{\text{sum}}(P)}{\log(1 + P/\sigma^2)} \ . \tag{7}$$

Note that Γ is defined only when the limit on the right side exists.

III. HIGH SNR

Consider the following achievable strategy. Each user m constructs a Gaussian random codebook at power P and rates R_m. Upon decoding, each user uses the codeword it sent across the channel as side-information to subtract off its own signal. Standard arguments (see, e.g. [8, p.]) then show that the partial sum-rates

$$\sum_{m \neq \ell} R_m < \log(1 + (M-1)s) \tag{8}$$

are achievable. By summing over ℓ and dividing by $M - 1$, the sum-rate

$$\sum_{m=1}^{M} R_m < \frac{M}{M-1} \log(1 + (M-1)s) \tag{9}$$

is achievable. Since the codebooks that do not employ feedback, we will refer to this strategy as the *non-feedback codebook strategy*. It turns out this relatively simple strategy is optimal at high SNR in that it attains the pre-log capacity.

Theorem 1: The pre-log capacity is

$$\Gamma = \frac{M}{M-1} \ , \tag{10}$$

and is attained by Gaussian codebooks in which each decoder subtracts off its own signal.

The proof can be found in the Appendix.

IV. PERFORMANCE OF FEEDBACK STRATEGIES

As we have seen at high SNR, encoders can achieve the pre-log capacity by ignoring the channel outputs. In general, however, more intricate strategies have the potential to perform better. To highlight this potential benefit, we will restrict our attention to the special case of a common output in this section, in which $Z_\ell(i) = Z(i)$ for all ℓ (or equivalently, $Y(i) = Y_\ell(i)$).

Note that each decoder has the option of ignoring its message side-informatio. Thus, a common output multiway channel can employ any strategy achievable via a multiple access channel (MAC) with feedback. A natural strategy is then Kramer's Fourier MEC [5]. Fourier MEC generalizes Ozarow's strategy [9] to a class of interference networks. For the M-user MAC, the strategy attains the sum-rate capacity under a per-symbol power constraint [6]. For our purposes, it is sufficient to consider the following result.

Proposition 1: For the M-user MAC, Fourier MEC achieves the rates (see e.g. [5])

$$\sum_{i=1}^{M} R_i < \log(1 + M\lambda s) \ , \tag{11}$$

where $1 \leq \lambda \leq M$ is the unique solution to

$$(M\lambda s + 1)^{M-1} = (\lambda(M - \lambda)s + 1)^M \ . \tag{12}$$

As mentioned above, these rates are automatically achievable for the common output M-user multiway channel that we consider in this section.

Note the sum-rate given by Fourier MEC in equation (11) does not attain the pre-log capacity stated in Theorem 1. However, Fourier MEC does provide a power gain for each user by correlating the channel inputs of the users. This allows it to achieve rates higher than the non-feedback codebook strategy for lower SNR.

One may be tempted to believe that the performance of Fourier MEC improves by using the channel input signals to refine the message estimate. While tedious to argue, one can show this does not increase the asymptotic rate of Fourier MEC. The problem stems from the fact that correlated channel inputs diminish the benefit the side-information. The key to improving the rate is to resolve this tension.

We now present a strategy to do just this, which uses feedback to construct an auxiliary channel. The strategy is a *super-letter strategy* in the sense that we group channel inputs into units of size k. Each codeword in our super-letter codebook

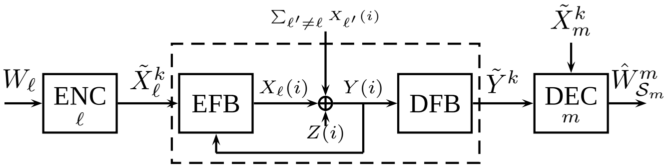

Fig. 2. Schematic of the super-letter strategy. At the encoder for user ℓ, each super-letter $\tilde{X}_\ell^k$ input gets converted into an input to the actual channel using a based on a modified Fourier MEC feedback strategy for k times steps, after which the feedback decoder sends its channel outputs for those k time steps as the vector $\tilde{Y}^k$ to each decoder m, which uses this and its side information about its own super-letter inputs to perform jointly typical decoding of the other messages.

is generated independently in n/k subblocks, and each subblock of length k– denoted $\tilde{X}_\ell^k[m]$– is generated independently and according to a jointly circular Gaussian vector distribution $\mathcal{N}_c(\vec{0}, P\mathbb{I})$. We will use the notation $\tilde{X}_\ell[m,i]$ to denote the i-th component of the m-th superletter for user ℓ.

Each super-letter outputted by the super-letter encoder enters a feedback encoder, which uses the current super-letter and channel outputs to construct inputs to the actual channel. A feedback decoder then groups the channel outputs into units of size k and outputs this to the super-letter decoder. The super-letter encoder and decoder treat the actions of the feedback encoder and decoder as a black box, and the decoder performs jointly typical decoding of the messages using its super-letter channel inputs as side information.

Specifically, each channel input is constructed as follows. For $q = mk$,

$$X_\ell(q{+}1) = \text{FMEC}^M_{\ell,1}\big(\tilde{X}_\ell[m,1])\big) \tag{13}$$

$$X_\ell(q{+}2) = \sqrt{\alpha}\text{FMEC}^M_{\ell,2}\big(\tilde{X}_\ell[m,1],Y(q{+}1)\big){+}\sqrt{1{-}\alpha}\tilde{X}_\ell[m,2] \tag{14}$$

$$\vdots$$

$$X_\ell(q{+}k) = \sqrt{\alpha}\text{FMEC}^M_{\ell,k}\big(\tilde{X}_\ell[m,1],Y_{q+1}^{q+k-1}\big){+}\sqrt{1{-}\alpha}\tilde{X}_\ell[m,k]\ , \tag{15}$$

where

$$\text{FMEC}^M_{\ell,1}(X) = \sqrt{\frac{P_\ell(1)}{E[X^2]}}X$$

$$\text{FMEC}^M_{\ell,i}(X,Y) = e^{j\frac{2\pi(\ell-1)(i-1)}{M}}\sqrt{\frac{P_\ell(i)}{D_{X|Y}}}(X - E[X|Y])$$

$$D_{X|Y} = E(X - E[X|Y])^2\ .$$

By appropriately choosing $P_\ell(i) \leq P$ for $i = 1, \ldots, M$ (see e.g. [5, p. 1435, Appendix C]), for $P_\ell(i) = P$ during time steps $i = M+1, \ldots, k$, $X_1(q+i), X_2(q+i), \ldots, X_M(q+i)$ are zero-mean jointly Gaussian with covariance matrix

$$\Sigma = P\begin{pmatrix} 1 & \tilde{\rho} & \cdots & \tilde{\rho} \\ \tilde{\rho} & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \tilde{\rho} \\ \tilde{\rho} & \cdots & \tilde{\rho} & 1 \end{pmatrix}\ , \tag{16}$$

where $(M-1)\tilde{\rho} = \tilde{\lambda}-1$, $\tilde{\lambda} = \alpha(\lambda^*-1)+1$, and $1 \leq \lambda^* \leq M$ is the unique solution to

$$\left(1 + \lambda^* M\frac{\alpha P}{M(1-\alpha)P+\sigma^2}\right)^{M-1} = \left(1 + \lambda^*(M-\lambda^*)\frac{\alpha P}{M(1-\alpha)P+\sigma^2}\right)^{M}\ . \tag{17}$$

We now use $0 \leq \alpha \leq 1$ as a parameter to adjust the level to which we correlate the channel inputs. Since λ^* is the solution of a polynomial equation with coefficients continuous in α, $\lambda^*(\alpha)$ is a continuous function of α, and thus $\tilde{\lambda}(\alpha)$ is a continuous function of α such that $1 \leq \tilde{\lambda}(\alpha) \leq \lambda^*(1)$. Thus, for $\lambda \overset{\text{def}}{=} \lambda^*(1) \geq M/2$, we will select α^* such that $\tilde{\lambda}(\alpha^*) = M/2$. Otherwise, we set $\alpha^* = 1$. For notational convenience, we will suppress this α dependence.

Then by joint typicality arguments on these blocks of length k and by targeting symmetric rates, one can show the following sum-rate is achievable:

$$\frac{M}{M-1}k^{-1}I(\tilde{X}^k_{\mathcal{S}_\ell}; Y^k|\tilde{X}^k_\ell)$$

$$= \frac{M}{M-1}k^{-1}\sum_{i=1}^{k} I(\tilde{X}^k_{\mathcal{S}_\ell}; Y(i)|Y^{i-1}, \tilde{X}^k_\ell) \tag{18}$$

$$= \frac{M}{M-1}k^{-1}\sum_{i=1}^{k} h(Y(i)|Y^{i-1}, X_\ell(i), \tilde{X}^k_\ell) - h(Y(i)|Y^{i-1}, X_\ell(i), \tilde{X}^k_1, \ldots, \tilde{X}^k_M) \tag{19}$$

$$= \frac{M}{M-1}k^{-1}\sum_{i=1}^{k} I(X_{\mathcal{S}_\ell}(i); Y(i)|Y^{i-1}, X_\ell(i), \tilde{X}^k_\ell) \tag{20}$$

$$\geq \frac{M}{M-1}\log\left(1 + \tilde{\lambda}(M-\tilde{\lambda})P/\sigma^2\right) - \frac{o(k)}{k} \tag{21}$$

$$= \begin{cases} \frac{M}{M-1}\log\left(1 + M^2/4P/\sigma^2\right) - \frac{o(k)}{k} & \lambda \geq M/2 \\ \frac{M}{M-1}\log\left(1 + \lambda(M-\lambda)P/\sigma^2\right) - \frac{o(k)}{k} & \text{o.w.} \end{cases}\ , \tag{22}$$

where $\mathcal{S}_\ell = \{1, \ldots, M\} - \{\ell\}$, and (21) since all random variables are jointly Gaussian by construction and by evaluating the determinants of their covariance matrices. This allows us to establish the following.

Theorem 2: There exists a choice of (α, k) in the super-letter strategy to achieves sum-rates arbitrarily close to

$$\sum_{k=1}^{M} R_k < \begin{cases} \frac{M}{M-1}\cdot\log\left(1 + \frac{M^2}{4}s\right) & \lambda \geq M/2 \\ \frac{M}{M-1}\cdot\log\left(1 + \lambda(M-\lambda)s\right) & \text{o.w.} \end{cases}\ , \tag{23}$$

where $1 \leq \lambda \leq M$ is the unique solution of (12).

Note that $\alpha = 0$ corresponds to the non-feedback codebook strategy considered in Section III, and $\alpha = 1$ corresponds to the Fourier MEC strategy. We now show that above a certain SNR threshold, the super-letter strategy achieves the sum-rate capacity.s

Theorem 3: The sum-rate capacity is upper bounded by

$$\sum_{k=1}^{M} R_k \leq \frac{M}{M-1}\log\left(1 + \frac{M^2}{4}s\right)\ , \tag{24}$$

Proof: Note that (34) is a valid upper bound on the sum-rate, and optimizing over ρ gives the result. ■

Figures 3 and 4 show the performance of the strategies in the high and intermediate SNR regimes; they reveal a clear advantage to using the feedback-based super-letter strategy.

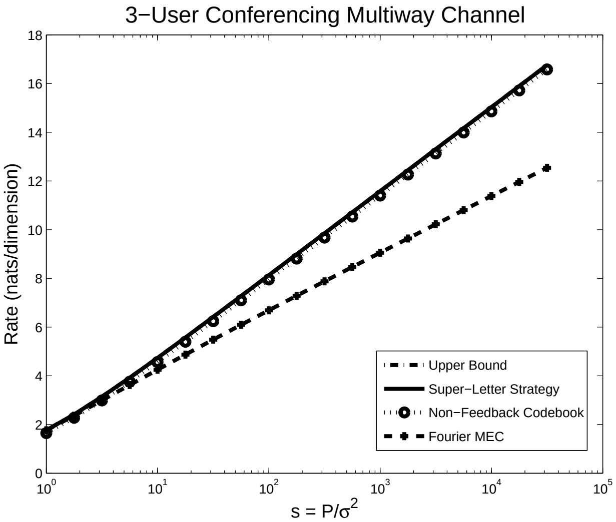

Fig. 3. Plot of the sum-rate for the strategies in the high SNR region for 3 users. In this regime, the non-feedback codebook outperforms Fourier MEC (see e.g. Theorem 1), but the super-letter strategy meets the upper bound exactly, corresponding to Theorems 2 and 3.

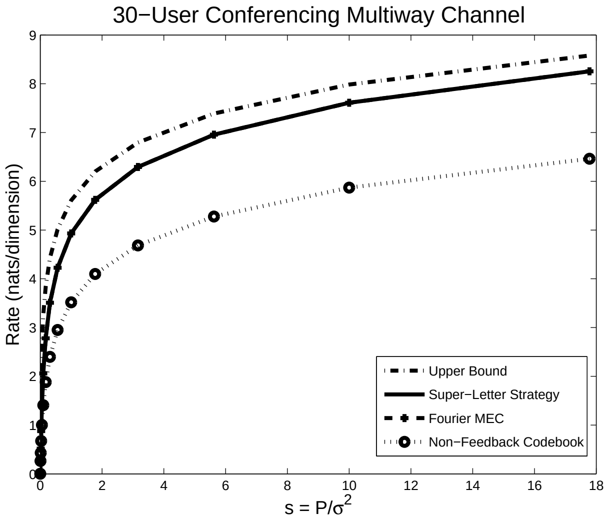

Fig. 4. Plot of the sum-rate for the achievable strategies in an intermediate SNR region for 30 users. In this region, Fourier MEC outperforms the non-feedback codebook and achieves the same rates as the super-letter strategy.

V. DISCUSSION AND EXTENSIONS

This paper considers a channel model in which feedback is implicit in the channel outputs at each user. While a non-feedback codebook strategy achieves the pre-log capacity at high SNR, we demonstrate that for the case of a common output, there is a feedback-based super-letter strategy that dominates the non-feedback codebook strategy in terms of sum-rate. The question remains as to how these strategies generalize to the case in which the noise at each user is different. By adapting truncated linear strategies of the kind considered in [10], one may be able to generalize the strategy beyond the common output case.

Another issue to consider is whether one can tighten the upper bound in Theorem 3 for all SNR. Hekstra and Willems derived dependence-balance bounds for common output two-way channels [12], and these have been used to show that under a per-symbol power constraint, i.e. $E|X_k(i)|^2 \le P$ for all i and k, Fourier MEC attains the sum-rate capacity of an M-user MAC [6]. If one carries through those arguments for the common output M-user multiway channel (see Appendix), one obtains

$$\sum_{k=1}^{M} R_k \le \frac{M}{M-1} \log(1+\lambda(M-\lambda)s) \ , \tag{25}$$

where $\lambda = \min\{\lambda^*, M/2\}$ and $1 \le \lambda^* \le M$ is the unique solution to

$$(1+M\lambda^* s)^{M-1} = (1+\lambda^*(M-\lambda^*)s)^M \ . \tag{26}$$

These bounds match the achievable sum-rates given by the super-letter strategy in Theorem 2 for all SNR. An open problem is whether this extends to block power constraints.

As a final point we note that this work has focused on symmetric powers and noises. Relaxing these assumptions is also the subject of future work.

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APPENDIX A
PROOF OF THEOREM 1

Lemma 1: Let $\mathcal{S}_\ell = \{1,\ldots,M\} - \ell$. Then for all $\ell \in \{1,\ldots,M\}$, the set of achievable partial sum-rates $\sum_{k\neq\ell} R_k$ contains the region

$$\sum_{k\neq\ell} R_k \le I(X_{\mathcal{S}_\ell}; Y_\ell | X_\ell) \tag{27}$$

for all $\{X_{\ell'}\}_{\forall\ell'}$ with marginal distributions satisfying $E[|X_{\ell'}|^2] \le P$.

Proof: By standard arguments, the partial sum-rate $\sum_{k\neq\ell} R_k$ can be bounded as follows for all ℓ:

$$\begin{aligned} n\sum_{k\neq\ell} R_k &\le \sum_{i=1}^{n} I(\{W_{\ell'}\}_{\ell'\neq\ell}; Y_\ell(i)|Y_\ell^{i-1}, W_\ell) + o(n) && (28)\\ &\le \sum_{i=1}^{n} h(Y_\ell(i)|X_\ell(i), Y_\ell^{i-1}, W_\ell) \\ &\qquad - h(Y_\ell(i)|Y_\ell^{i-1}, \{X_{\ell'}(i), W_{\ell'}\}_{\forall\ell'}) + o(n) && (29)\\ &\le nI(X_{\mathcal{S}_\ell}; Y_\ell|X_\ell) + o(n) \ , && (30) \end{aligned}$$

where $X_{\mathcal{S}_\ell} = X_{\mathcal{S}_\ell}(T)$, $Y_\ell = Y_\ell(T)$, $X_\ell(T)$, and T is defined to be uniform over $\{1, \ldots, n\}$. Letting $n \to \infty$ completes the result. ■

By Lemma 1,

$$\sum_{m \neq \ell} R_m \leq I(X_{\mathcal{S}_\ell}; Y_\ell | X_\ell) \ . \tag{31}$$

By a maximum entropy argument and summing over ℓ

$$\sum_{\ell=1}^{M} \sum_{m \neq \ell}^{M} R_m \leq \sum_{\ell=1}^{M} \log \frac{V_{Y|X_\ell}}{\sigma^2} \ . \tag{32}$$

Since $\log V_{Y|X_\ell}$ is concave in K_{Y,X_ℓ} [8, Theorem 16.9.1, p. 506], one can maximize the sum-rate by making the covariance matrices K_{Y,X_ℓ} the same. It is straightforward to show that this is equivalent to requiring the input covariance matrix to be of the form

$$K_X = P \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} , \tag{33}$$

which implies that

$$(M-1) \sum_{k=1}^{M} R_k \leq \\ M \log\left(1 + (M-1)(1 + (M-1)\rho)(1-\rho)s\right) \ . \tag{34}$$

By dividing both sides by $(M-1)\log(s+1)$ and taking the limit as $s \to \infty$, one arrives at the upper bound

$$\Gamma \leq \frac{M}{M-1} \ . \tag{35}$$

Recall the non-feedback codebook strategy achieves the same pre-log, thereby completing the proof.

APPENDIX B
DEPENDENCE-BALANCE

We use the remainder of the appendix to outline the argument that leads to the extension considered in the discussion.

Letting $\mathcal{S}_\ell = \{1, \ldots, M\} - \{\ell\}$, it is straightforward to show that

$$n \sum_{k \neq \ell} R_k \leq \sum_{i=1}^{n} I(\{X_{s,i}\}_{s \in \mathcal{S}_\ell}; Y_i | X_{\ell,i}, Y^{i-1}) + o(n) \ , \tag{36}$$

Note that under a per-symbol power constraint, one can optimize the optimal channel inputs at each time instance by choosing them to be Gaussian. By standard arguments (see e.g. [12], [6]), one can derive a dependence-balance bound, which shows that any codebook trivially satisfies the inequality

$$0 \leq \sum_{i=1}^{n} \Big[-I(X_{1,i}, \ldots, X_{M,i}; Y_i | Y^{i-1}) \\ + (M-1)^{-1} \sum_{\ell=1}^{M} I(\{X_{s,i}\}_{s \in \mathcal{S}_\ell}; Y_i | X_{\ell,i}, Y^{i-1}) \Big] \tag{37}$$

which in the Gaussian setting simplifies to

$$0 \leq \frac{1}{n} \prod_{i=1}^{n} \log \frac{\prod_{k=1}^{M} V_{Y_i|Y^{i-1},X_{k,i}}}{(V_{Y_i|Y^{i-1}})^{M-1} V_{Y_i|Y^{i-1},X_{1,i},\ldots,X_{M,i}}} \ . \tag{38}$$

While not particularly meaningful in the multi-letter setting, the dependence-balance bounds become become useful in simplifying these expressions. Standard concavity arguments about the input covariance matrices argued in [6] allow one to simplify the multi-letter inequalities in (36) and (38) to

$$\sum_{k=1}^{M} R_k \leq \frac{M}{M-1} \log(1 + \lambda(M-\lambda)s) \ , \tag{39}$$

where $1 \leq \lambda \leq M$ satisfies

$$(1 + M\lambda s)^{M-1} \leq (1 + \lambda(M-\lambda)s)^{M} \ . \tag{40}$$

One can show that the right side is concave, larger than the left side for $\lambda = 1$ and smaller at $\lambda = M$. Since the left side increases in λ, they intersect at exactly one point, below which all choices of λ are valid. Theorem 3 implies that if this intersection point is above $M/2$, choosing $\lambda = M/2$ maximizes the sum-rate. Otherwise, the sum-rate is maximized at the point of intersection.

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